Recap

For linear transformation $\varphi : V \to W$

- $\ker(\varphi) = \{ v \in V \mid \varphi(v) = 0 \} \subseteq V$
- $\operatorname{im}(\varphi) = \{ \varphi(v) \mid v \in V \} \subseteq W$

$$\dim(\operatorname{im}(\varphi)) + \dim(\ker(\varphi)) = \dim(V)$$

For linear operator $\varphi : V \to V$

- $\lambda \in \mathbb{F}$ is an eigenvalue if $\exists\, v \in V \setminus \{0_V\}$ s.t.

$$\varphi(v) = \lambda \cdot v$$

- If $\varphi(v_1) = \lambda_1 \cdot v_1, \ \dots, \ \varphi(v_k) = \lambda_k \cdot v_k$ for distinct $\lambda_1 \dots \lambda_k$ then $v_1, \dots, v_k$ are LI.

Diagonalizable operators

$\varphi : V \to V$ is said to be diagonalizable if $\exists$ basis $\mathcal{B}$ of V

s.t. every $v \in \mathcal{B}$ is an eigenvector of φ.

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$

others ?

Ex: Check that $\varphi_{\text{left}} : \text{Fib} \to \text{Fib}$ is diagonalizable.

Inner Products

Vector space V over $\mathbb{F}$ (= $\mathbb{R}$ or $\mathbb{C}$ for this part)

$\mu : V \times V \to \mathbb{F}$ is an inner product if

- $\forall u \in V$ $\quad \mu(u, \cdot) : V \to \mathbb{F}$ is a linear transformation
- $\forall u, v \in V$ $\quad \mu(u,v) = \overline{\mu(v,u)}$
- $\forall v \in V$ $\quad \mu(v,v) \in \mathbb{R}_{\geq 0}$ and $\mu(v,v) = 0$ iff $v = 0_V$

Examples

- $V = \mathbb{R}^2$, $\quad \mu((x_1,x_2),(y_1,y_2)) = x_1 \cdot y_1 + x_2 \cdot y_2$
- $V = \mathbb{C}^2$, $\quad \mu((x_1,x_2),(y_1,y_2)) =$
- $V = \mathbb{R}^2$ $\quad \mu((x_1,x_2),(y_1,y_2)) = 2x_1y_1 + x_2y_2 + \frac{1}{2}(x_1y_2 + x_2y_1)$

Other examples

- $V = C([-1,1], \mathbb{R})$

$$\mu(f,g) = \int_{-1}^{1} f(x)\, g(x)\, dx$$

$$\mu(f,g) = \int_{-1}^{1} \frac{f(x)\, g(x)}{\sqrt{1-x^2}}\, dx$$

Ex: For $V = \text{Fib}$, for $c > 4$

$$\mu(f,g) = \sum_{n=0}^{\infty} \frac{f(n)\, g(n)}{c^n} \text{ is an inner product}$$

Conjugate-linearity

Recall $\mu(u, \cdot) : V \to \mathbb{F}$ is a linear transformation

- $\forall v \in V$ $\mu(\cdot, v) : V \longrightarrow \mathbb{F}$ is conjugate-linear i.e.,

  - $\mu(u_1 + u_2, v) = \mu(u_1, v) + \mu(u_2, v)$

  - $\mu(c \cdot u, v) = \bar{c} \cdot \mu(u, v)$

Proof: $\mu(c \cdot u, v) = \overline{\mu(v, c \cdot u)}$

$$= \overline{c \cdot \mu(v, u)}$$

$$= \bar{c} \cdot \overline{\mu(v, u)}$$

$$= \bar{c} \cdot \mu(u, v)$$

Cauchy-Bunyakovsky-Schwarz inequality

Cauchy Bunyakovsky Schwarz

- For any $u, v \in V$

$$|\langle u,v\rangle|^2 \leq \underbrace{\langle u,u\rangle}_{\|u\|^2} \cdot \underbrace{\langle v,v\rangle}_{\|v\|^2}$$

Proof: $w = u - t\cdot v$

$$\langle w,w\rangle \geq 0 \iff \langle u,u\rangle + t\cdot\bar{t}\,\langle v,v\rangle - \left(t\cdot\langle u,v\rangle + \overline{t\cdot\langle u,v\rangle}\right) \geq 0$$

choosing $t = \overline{\langle u,v\rangle} / \langle v,v\rangle$ gives

$$\langle u,u\rangle + \frac{|\langle u,v\rangle|^2}{\langle v,v\rangle} - 2\,\frac{|\langle u,v\rangle|^2}{\langle v,v\rangle} \geq 0$$

$$\iff \langle u,u\rangle - \frac{|\langle u,v\rangle|^2}{\langle v,v\rangle} \geq 0 \iff \langle u,u\rangle\cdot\langle v,v\rangle \geq |\langle u,v\rangle|^2 .$$

Ex: When is this tight?

Distances from inner products

- Define $\|u\| = \sqrt{\langle u, u \rangle}$ } Defined for any inner product

- $\|u-v\|$ defines a distance i.e.

Ex: $\forall u, v, w \in V \quad \|u-v\| + \|v-w\| \geq \|u-w\|$

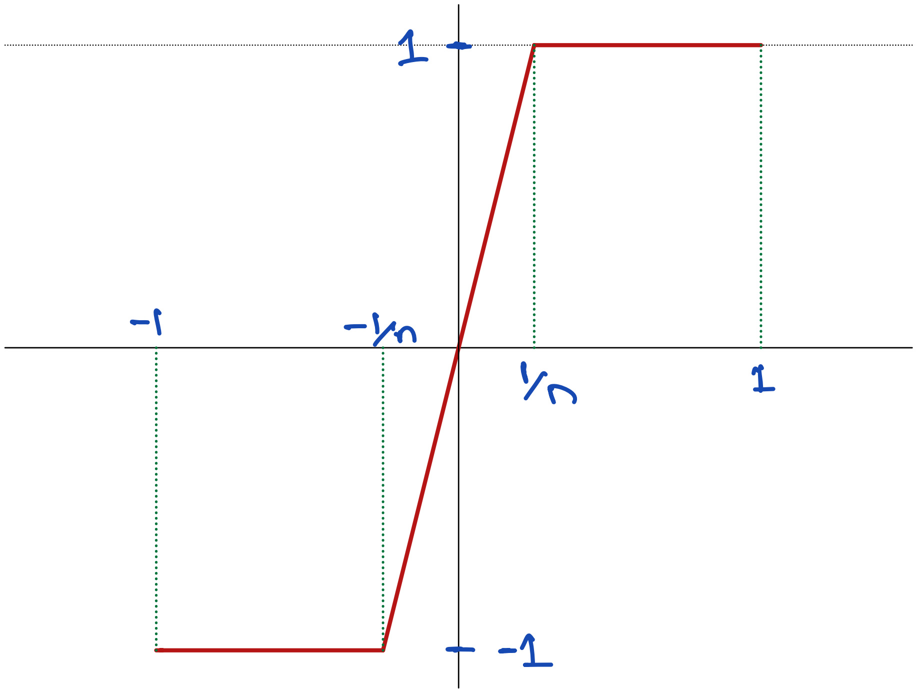
(Triangle inequality)

- Can now talk of distances, convergence, limits ...

Hilbert Spaces

- Limits may exist, but are they still in V ?

e.g. $V = \mathcal{C}([-1,1], \mathbb{R})$

$f_n =$ 

$\lim_{n \to \infty} f_n = ?$

Hilbert Space : Closure of inner-product space under limits

Orthogonality and Orthonormality

- u, v are orthogonal (under μ) if $\langle u, v \rangle_\mu = 0$.
- S is a set of mutually orthogonal vectors if

  $$\langle u, v \rangle = 0 \quad \forall\, u, v \in S \quad u \neq v$$

- $S \subseteq V$ is orthonormal if
  - $\langle u, v \rangle = 0 \quad \forall\, u, v \in S \quad u \neq v$
  - $\|u\| = 1 \quad \forall\, u \in S$

Examples

- $V = \mathbb{R}^2$ with $\langle x, y \rangle = x_1 y_1 + x_2 y_2$ $\quad S = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$
- $V = \mathbb{R}^2$ with $\langle x, y \rangle = 2x_1 y_1 + x_2 y_2 + \frac{1}{2} x_1 y_2 + \frac{1}{2} x_2 y_1$

Orthogonality implies linear independence

- Let $S \subseteq V \setminus \{0_V\}$ be a set of mutually orthogonal vectors. Then S is LI.

Proof: Say $c_1 \cdot v_1 + \cdots + c_n \cdot v_n = 0$ for $v_1, \dots, v_n \in S$

$$\langle v_1, c_1 \cdot v_1 + \cdots + c_n \cdot v_n \rangle = 0$$

$$\Rightarrow c_1 \cdot \langle v_1, v_1 \rangle + 0 \cdots + 0 = 0 \Rightarrow c_1 = 0$$

Similarly $c_2, \dots, c_n = 0$.

Gram - Schmidt orthogonalization

▷ Given $\{v_1, \dots, v_n\}$ LI, $\exists \{w_1 \dots w_n\}$ orthonormal s.t.

$$\mathrm{Span}(\{w_1, \dots, w_n\}) = \mathrm{Span}(\{v_1, \dots, v_n\})$$

Proof: Induction on $k \in \{1, \dots, n\}$

$k = 1$

$$w_1 = \frac{v_1}{\|v_1\|}$$

$k \to k+1$

$\exists \{w_1, \dots, w_k\}$ orthonormal s.t.

$$\mathrm{Span}(\{w_1 \dots w_k\}) = \mathrm{Span}(\{v_1 \dots v_k\})$$

$$\tilde{w}_{k+1} = c_1 \cdot w_1 + \dots + c_k \cdot w_k + v \quad \text{(say)}$$

$$\langle w_i, \tilde{w}_{k+1} \rangle = 0 \Rightarrow c_i = -\langle w_i, v \rangle$$

$$w_{k+1} = \frac{\tilde{w}_{k+1}}{\|\tilde{w}_{k+1}\|}$$

Orthonormal bases

- Every finite dimensional inner product space has an orthonormal basis.

<u>Proof:</u> Basis $\{v_1, \ldots v_n\}$

$$\{v_1, \ldots, v_n\} \ \big\downarrow \text{ Gram-Schmidt} \ \{w_1, \ldots w_n\}$$

Ex: Find an orthonormal basis for $V = \mathbb{R}^2$ with .

$$\langle x, y\rangle = (x_1 \ \ x_2)\begin{pmatrix} 2 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- Hilbert spaces also have (countable) orthonormal bases if we allow infinite sums (Hilbert bases)